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Classical Partition Function for Non-Relativistic Gravity

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Abstract: We considered the canonical gravitational partition function Z associated to the classical Boltzmann–Gibbs (BG) distribution $\frac{e^{-\beta H}}{\mathcal{Z}}$. It is popularly thought that it cannot be built up because the integral involved in constructing Z diverges at the origin. Contrariwise, it was shown in (Physica A 497 (2018) 310), by appeal to sophisticated mathematics developed in the second half of the last century, that this is not so. Z can indeed be computed by recourse to (A) the analytical extension treatments of Gradshteyn and Rizhik and Guelfand and Shilov, that permit tackling some divergent integrals and (B) the dimensional regularization approach. Only one special instance was discussed in the above reference. In this work, we obtain the classical partition function for Newton's gravity in the **four** cases that immediately come to mind.

Keywords: partition functions; analytical extensions; guelfand's and gradshteyn's; classical gravity

MSC: 32A70; 46N55; 82B03; 82B05

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1. Introduction

This paper is a continuation and generalization of [1]. It involves mathematical ideas that were fully explored there (and references therein), in which a canonical ensemble at the temperature T was concocted for a two-particle gravitation system and fully analyzed. For brevity's sake, the accompanying (involved) math will not be repeated here. It is advisable to have [1] at hand in trying to follow our discussion below. A very important concept is that of generalized dimensionally regularized partition function Z, which we abbreviate as $GDR[\mathcal{Z}$. Dimensional regularization works in ν dimensions. We call ν_0 the actual physical dimension. We use the common notation $\beta = 1/(k_B T)$, with T as the temperature. To simplify numerical computations, we set $k_B = 1$, and also the gravitational constant G equal to unity. The domain of integration is called D-.

An important point that will emerge below is that of the behavior of the partition function $\mathcal{Z}$ as a function of the inverse temperature β. $\mathcal{Z}$ is a sum or integral of terms of the form $\exp[-\beta E_M]$, with E_m some energy. Thus, the second derivative $d\mathcal{Z}/d\beta^2$ is positive. Thus, its curvature, when plotted against β, *cannot change*.

Self-gravitating systems exhibit peculiarities that might perhaps defy ordinary common sense. We highlight here the following: [2]

(1) As gravitational binding gets tighter, the kinetic energy augments and so does, as a consequence, the temperature;
(2) As gravitational binding gets weaker, the kinetic energy decreases, and as a consequence, the temperature diminishes;
(3) The specific heat becomes negative if the system can freely expand or contract.

2. Basic Instance: Partition Function for the Distance-Case $0 \leq r < \infty$

The first case is the most important one. Conceptually, it is the most complex of the four to be addressed. Algebraically, however, it is the simplest instance. We deal with the gravitational interaction between two masses m and M. The partition function is then:

$$\mathcal{Z} = \mathcal{Z}_{\nu_0} = (-1)^{\nu_0} GDR[\mathcal{Z}_\nu]_{\nu=\nu_0}, \tag{1}$$

and $\mathcal{Z}_\nu$ is given by classical phase space (ν dimensions):

$$\mathcal{Z}_\nu = \int\limits_D e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} d^\nu x d^\nu p. \tag{2}$$

This quantity was already used for three dimensions in [1]. We then have:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \iint\limits_0^\infty (rp)^{\nu-1} e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} dr\, dp. \tag{3}$$

By appeal to the relation below, employed in [1]:

$$\int\limits_0^\infty x^{\nu-1} e^{\frac{\beta}{x}} dx = \cos(\pi\nu)\beta^\nu \Gamma(-\nu), \tag{4}$$

we obtain:

$$\mathcal{Z}_\nu = \left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \frac{\cos(\pi\nu)}{2} (\beta GmM)^\nu \left(\frac{2m}{\beta}\right)^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)\Gamma(-\nu). \tag{5}$$

To apply GDR, we introduce $f(\nu)$:

$$f(\nu) = \frac{2\pi^\nu}{\Gamma\left(\frac{\nu}{2}\right)} \frac{(\beta GmM)^n u}{\nu(\nu-1)(\nu-2)} \left(\frac{2m}{\beta}\right)^{\frac{\nu}{2}}, \tag{6}$$

and in three dimensions:

$$\mathcal{Z}_\nu = f(\nu)\Gamma(3-\nu). \tag{7}$$

Following [1], we Laurent expand around $\nu = 3$ to obtain:

$$\mathcal{Z}_\nu = -\frac{2}{3\sqrt{\pi}} \frac{(2\pi^2\beta G^2 m^3 M^2)^{\frac{3}{2}}}{(\nu-3)} - \frac{1}{3\sqrt{\pi}} (2\pi^2\beta G^2 m^3 M^2)^{\frac{3}{2}} \times$$
$$\left[\ln\left(8\pi^2\beta G^2 m^3 M^2\right) + 3\boldsymbol{C} - \frac{17}{3}\right] + \sum_{n=1}^\infty a_n(\nu-3)^n. \tag{8}$$

Therefore:

$$\mathcal{Z} = \frac{1}{3\sqrt{\pi}} (2\pi^2\beta G^2 m^3 M^2)^{\frac{3}{2}} \left[\ln\left(8\pi^2\beta G^2 m^3 M^2\right) + 3\boldsymbol{C} - \frac{17}{3}.\right] \tag{9}$$

Remind that the mean energy $< \mathcal{U} >= -\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \beta}$. Accordingly, we obtain:

$$< \mathcal{U} >= -\frac{\sqrt{2}\pi^{5/2}\left(G^2 m^3 M^2 \beta\right)^{3/2} \left(3C + \log\left(8\pi^2 G^2 m^3 M^2 \beta\right) - 5\right)}{\beta \mathcal{Z}}. \tag{10}$$

For the specific heat C, we use the definition $C = \frac{\partial < \mathcal{U} >}{\partial T}$, and we have:

$$C = \frac{\pi^{5/2} G^4 m^6 M^4 \beta \left(3C + \log\left(8\pi^2 G^2 m^3 M^2 \beta\right) - 3\right)}{\sqrt{2}\sqrt{G^2 m^3 M^2 \beta} T \mathcal{Z}}$$

$$-\frac{\beta}{T}\left[\frac{\sqrt{2}\pi^{5/2}\left(G^2m^3M^2x\right)^{3/2}\left(3C+\log\left(8\pi^2G^2m^3M^2x\right)-5\right)}{\beta\mathcal{Z}}\right]^2. \tag{11}$$

We express a word of caution regarding the plotting the above relations. They involve big quantities like m and M and also very small ones like G and k_B. In order to make sense of the associated computational mess, we are forced to appeal to a simple scenario with $m = M = G = k_B = 1$ and set the horizontal coordinate to $x = 1/T$. Notice that we will be dealing with gigantic temperatures in the order of 10^{22} Kelvin. See Figures 1–3.

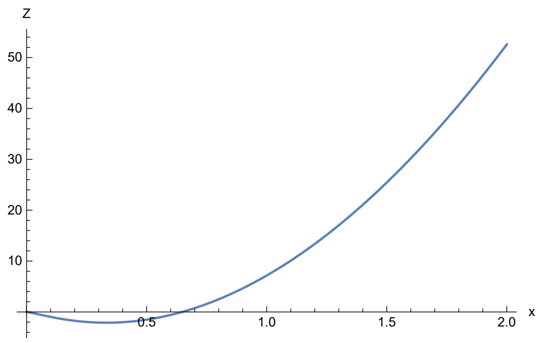

Figure 1. $m = M = G = k_B = 1$, and $x = 1/T$. Plot of $\mathcal{Z}(x)$. Here, $x = \beta$, $G = m = M = 1$. For small x, we see that Z is negative. These x-values are, of course, unphysical. Z grows as T diminishes.

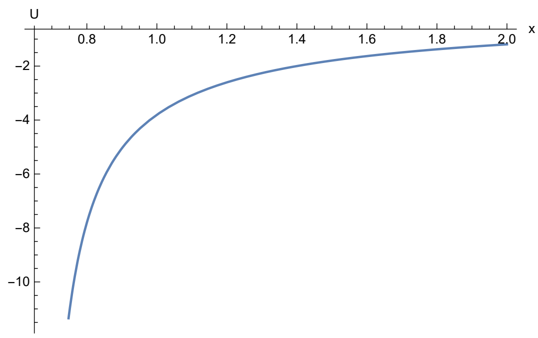

Figure 2. $m = M = G = k_B = 1$, and $x = 1/T$. Plot of $< \mathcal{U}(x) >$. At large temperatures (near the origin), the binding and kinetic energies, as well as the temperature are large, as prescribed by item (1) of the introduction. As T decreases, so does the binding and kinetic energies, as indicated by item (2) of the Introduction.

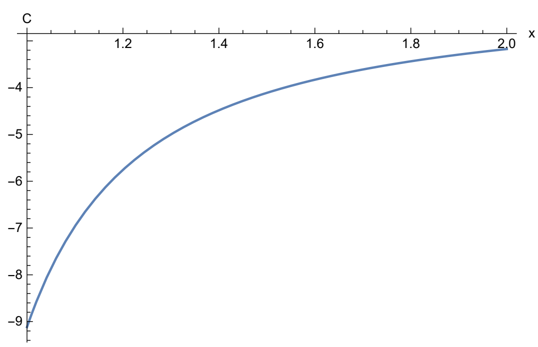

Figure 3. Plot of the specific heat $C(x)$. Here, $x = \beta$, $G = m = M = 1$. It is negative, as prescribed by item (3) of the Introduction. The specific heat tends to vanish as the temperature drops, as expected because of the third law of thermodynamics, that, remarkably enough, is obeyed classically here.

We see that our dimensionally regularized partition function predicts the expected gravitational behavior.

3. Partition Function for the Bound System $R_0 \leq r \leq R_1$

Now, suppose that we deal with two masses: one has mass M and radius R_0 and the second is a point mass m. Both are contained in a spherical box of radius R_1. Of course, in this paper, tidal forces are ignored. The system is bound by construction.

The partition function now turns out to be:

$$\mathcal{Z} = \left[\frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right]^2 \int_0^\infty dp \int_{R_0}^{R_1} dr (rp)^2 e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)}. \tag{12}$$

We appeal to the well-known result:

$$\int x^2 e^{\frac{a}{x}} dx = \frac{1}{6}\left[e^{\frac{a}{x}} x\left(a^2 + ax + 2x\right) - a^3 E_i\left(\frac{a}{x}\right)\right] + b. \tag{13}$$

Here, E_i is the integral exponential function [3] and b is an arbitrary constant. We obtain thus, in three dimensions:

$$\mathcal{Z} = \frac{\pi^{\frac{3}{2}} V_1}{2}\left\{e^{\frac{\beta GmM}{R_1}}\left[\left(\frac{\beta GmM}{R_1}\right)^2 + \frac{\beta GmM}{R_1} + 2\right] - \left(\frac{\beta GmM}{R_1}\right)^3 E_i\left(\frac{\beta GmM}{R_1}\right)\right\}$$
$$- \frac{\pi^{\frac{3}{2}} V_0}{2}\left\{e^{\frac{\beta GmM}{R_0}}\left[\left(\frac{\beta GmM}{R_0}\right)^2 + \frac{\beta GmM}{R_0} + 2\right] - \left(\frac{\beta GmM}{R_0}\right)^3 E_i\left(\frac{\beta GmM}{R_0}\right)\right\}, \tag{14}$$

where V_1 is the volume of a sphere of radius R_1 and V_0 is the volume of a sphere of radius R_0. For the mean energy, we have:

$$< \mathcal{U} >= -(1/(2R_1^3 R_0^3 \mathcal{Z}))\times$$
$$3\mathrm{GmM}\pi^{3/2}\Big(-V_1 R_1 R_0^3 e^{\frac{\mathrm{GmM}\beta}{R_1}}(R_1 + \mathrm{GmM}\beta) + R_1^3 V_0 R_0 e^{\frac{\mathrm{GmM}\beta}{R_0}}(R_0 + \mathrm{GmM}\beta) + V_1 R_0^3 \mathrm{GmM}^2$$
$$\beta^2 \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right) - R_1^3 V_0 \mathrm{GmM}^2 \beta^2 \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_0}\right)\Big), \tag{15}$$

and for the specific heat:

$$C = -\frac{3\pi^{3/2}\mathrm{GmM}^2\beta}{\mathcal{Z} T R_1^3 R_0^3}\times$$
$$\left(-V_1 R_1 R_0^3 e^{\frac{\mathrm{GmM}\beta}{b}} + R_1^3 V_0 R_0 e^{\frac{\mathrm{GmM}\beta}{R_0}} + V_1 R_0^3 \mathrm{GmM}\beta \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right) - \right.$$
$$\left. R_1^3 V_0 \mathrm{GmM}\beta \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_0}\right)\right)$$
$$-\frac{\beta}{T}\Big[(1/(2R_1^3 R_0^3 \mathcal{Z}))$$
$$3\mathrm{GmM}\pi^{3/2}\Big(-V_1 R_1 R_0^3 e^{\frac{\mathrm{GmM}\beta}{R_1}}(R_1 + \mathrm{GmM}\beta) + R_1^3 V_0 R_0 e^{\frac{\mathrm{GmM}\beta}{R_0}}(R_0 + \mathrm{GmM}\beta) + V_1 R_0^3 \mathrm{GmM}^2$$
$$\beta^2 \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right) - R_1^3 V_0 \mathrm{GmM}^2 \beta^2 \mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_0}\right)\Big)\Big]^2. \tag{16}$$

See Figures 4–6.

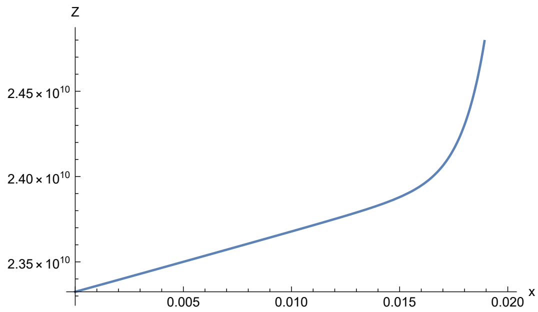

Figure 4. Plot of $\mathcal{Z}(x)$. Here, $x = \beta/R_1$, $R_0 = 1$, $R_1 = 1000$, $G = m = M = 1$.

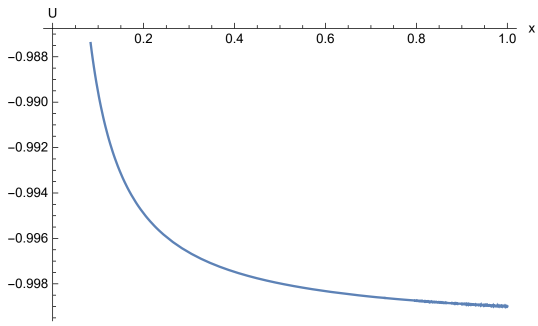

Figure 5. Plot of $< \mathcal{U}(x) >$. Here, $x = \beta/R_1$, $R_0 = 1$, $G = m = M = 1$. Note that the size of the container diminishes as x grows, so that the binding energy has to grow with x.

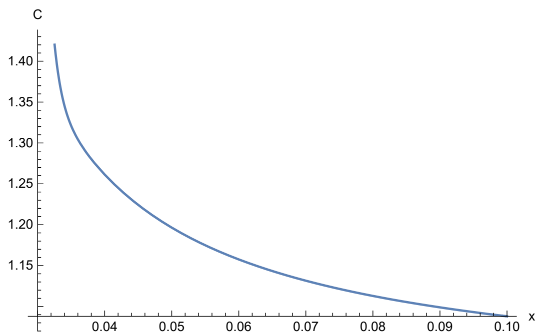

Figure 6. Plot of $C(x)$ versus x. Here, $x = \beta/R_1$, $R_0 = 1$, $G = m = M = 1$. The binding energy increases as β grows and the mean energy U is negative. Thus, the specific heat $= -\beta^2 \frac{dU}{d\beta}$ is positive. Notably enough, it tends to obey the third law in a classical scenario, as bound by construction.

4. Partition Function for $0 \leq r \leq R_1$

We now consider the case of two point masses m and M are enclosed in a container of radius R_1. The three-dimensional partition function is now:

$$\mathcal{Z} = -\left[\frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right]^2 \int_0^\infty dp \int_0^{R_1} dr (rp)^2 e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} \tag{17}$$

Evaluating the integral corresponding to the momenta, we arrive at:

$$\mathcal{Z} = -4\pi^{\frac{5}{2}} \left(\frac{2m}{\beta}\right)^{\frac{3}{2}} \int_0^{R_1} dr r^2 e^{\frac{\beta GmM}{r}}. \tag{18}$$

We see that the resulting integral displays a singularity at the origin. We evaluate this integral using the Guelfand [4] method for calculating integrals of powers of x, expanding the exponential in power series around the origin. We are led to:

$$\mathcal{Z} = -4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\sum_{n=0}^{\infty}\frac{(\beta GmM)^n}{n!}\int_0^{R_1} r^{2-n}dr. \tag{19}$$

We need here the result (see [4]):

$$\int_0^{R_1} r^{2-s}dr = \frac{R_1^{3-s}}{3-s} \quad ; \quad s \neq 3$$

$$\int_0^{R_1} r^{-1}dr = \ln R_1 \quad ; \quad s = 3. \tag{20}$$

Using it, we obtain:

$$\mathcal{Z} = -4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\left[(\beta GmM)^3\frac{\ln R_0}{3!} - 3\sum_{n=0;n\neq 3}^{\infty}\left(\frac{\beta GmM}{R_1}\right)^n\frac{R_1^3}{n!(n-3)}\right]. \tag{21}$$

Remember that we usually call C the Euler–Mascheroni constant. We now need a further result that is given in reference [5] and reads:

$$\sum_{s=0;s\neq 3}\frac{y^s}{s!(s-3)} = \frac{y^3}{3!}[\psi(4) - \ln|y| + E_1(y)] - \frac{e^y}{3!}[y^2 + y + 2], \tag{22}$$

so that we finally obtain:

$$\mathcal{Z} = -4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\left\{\frac{R_1^3 e^{\frac{\beta GmM}{R_1}}}{3!}\left[\left(\frac{\beta GmM}{R_1}\right)^2 + \frac{\beta GmM}{R_1} + 2\right] + \right.$$
$$\left.\frac{(\beta GmM)^3}{3!}\left[\ln(\beta GmM) - \psi(4) - E_i\left(\frac{\beta GmM}{R_1}\right)\right]\right\} \tag{23}$$

where ψ is the poly-gamma function. For the mean energy, one has:

$$<\mathcal{U}> = \frac{GmM\pi^{5/2}}{\mathcal{Z}}\left(2R_1^2 e^{\frac{\mathrm{GmM}\beta}{R_1}} + 2R_1\mathrm{GmM}\beta e^{\frac{\mathrm{GmM}\beta}{R_1}} - 3\mathrm{GmM}^2\beta^2 + 2C\mathrm{GmM}^2\beta^2 - \right.$$
$$\left. 2\mathrm{GmM}^2\beta^2\mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right) + 2\mathrm{GmM}^2\beta^2\log(\mathrm{GmM}\beta)\right), \tag{24}$$

and the specific heat reads:

$$C_v = -\frac{4GmM^2\pi^{5/2}\beta}{\mathcal{Z}T}\left(R_1 e^{\frac{\mathrm{GmM}\beta}{R_1}} - \mathrm{GmM}\beta + \gamma\mathrm{GmM}\beta - \right.$$
$$\left. \mathrm{GmM}\beta\mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right) + \mathrm{GmM}\beta\log(\mathrm{GmM}\beta)\right) -$$
$$\frac{\beta}{T}\left[\frac{PGmM\pi^{5/2}}{\mathcal{Z}}\left(2R_1^2 e^{\frac{\mathrm{GmM}\beta}{R_1}} + 2R_1\mathrm{GmM}\beta e^{\frac{\mathrm{GmM}\beta}{R_1}} - 3\mathrm{GmM}^2\beta^2 + 2\gamma\mathrm{GmM}^2\beta^2 - \right.\right.$$

$$2\mathrm{GmM}^2\beta^2\mathrm{Ei}\left(\frac{\mathrm{GmM}\beta}{R_1}\right)+2\mathrm{GmM}^2\beta^2\log(\mathrm{GmM}\beta)\Big)\Big]^2. \tag{25}$$

We plot below these equations in Figures 7–9.

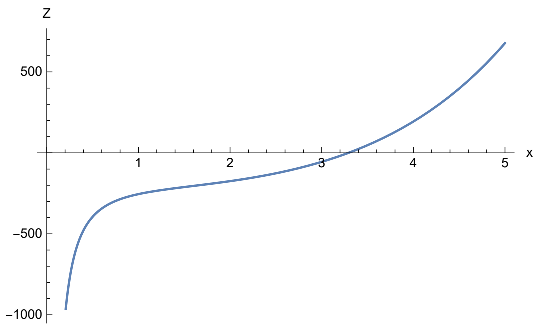

Figure 7. Plot of $\mathcal{Z}(x)$ versus $x=\beta$ for $G=m=R_1=M=1$. Results are unphysical for $x<3.35$. At the ensuing very high (big-bang like) temperatures (at which matter does not exist), $\mathcal{Z}(x)$ is negative, and thus unphysical.

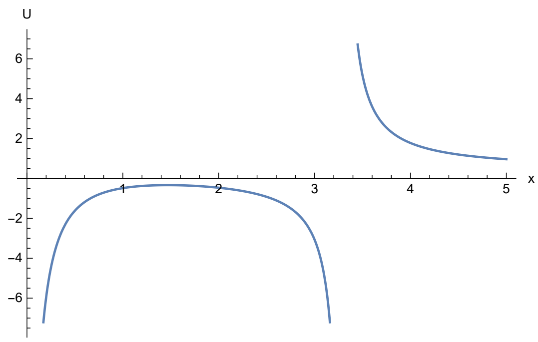

Figure 8. Plot of $<\mathcal{U}(x)>$ for $x=\beta$ $G=m=R_1=M=1$. Remember that the results are unphysical for $x<3.35$. The kinetic energy diminishes as x grows and so does the energy. However, R_1 is too large to allow for statistical bounding in the region here considered.

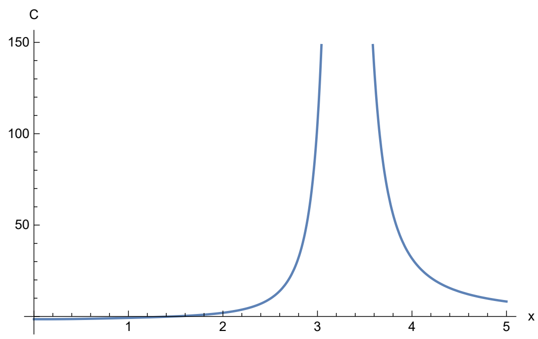

Figure 9. Plot of $C(x)$ for $x=\beta$. One has $G=m=R_1=M=1$. Again, the results are unphysical for $x<3.35$. The system tends to comply with the third law. It is unbound, since $C>0$.

5. Partition Function for $R_0 \leq r < \infty$

Finally, we confront the twin case of the precedent one (under a y to $1/y$ transform). We thus consider the case of a spherical mass M of radius R_0 interacting with a punctual mass m Accordingly, the distance between the two masses has a lower bound but no upper bound:

$$\mathcal{Z}=\left[\frac{2\pi^{\frac{3}{2}}}{\Gamma(\frac{3}{2})}\right]^2\int\limits_0^\infty dp\int\limits_{R_0}^\infty dr(rp)^2e^{-\beta\left(\frac{p^2}{2m}-\frac{GmM}{r}\right)} \tag{26}$$

Evaluating the momenta integral, we obtain:

$$\mathcal{Z} = 4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\int\limits_{R_0}^{\infty} dr r^2 e^{\frac{\beta GmM}{r}}. \tag{27}$$

This integral is divergent and can be evaluated as prescribed in reference [4], that is:

$$\mathcal{Z} = 4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\sum_{n=0}^{\infty}\frac{(\beta GmM)^n}{n!}\int\limits_{R_0}^{\infty} r^{2-n}dr, \tag{28}$$

leading to:

$$\int\limits_{R_0}^{\infty} r^{2-s}dr = -\frac{R_0^{3-s}}{3-s} \quad ; \quad s \neq 3$$
$$\int\limits_{R_0}^{\infty} r^{-1}dr = -\ln R_0 \quad ; \quad s = 3. \tag{29}$$

Using again the result (20), we obtain:

$$\mathcal{Z} = 4\pi^{\frac{5}{2}}\left(\frac{2m}{\beta}\right)^{\frac{3}{2}}\left\{\frac{(\beta GmM)^3}{3!}\left[\psi(4) + E_i\left(\frac{\beta GmM}{R_0}\right) - \ln(\beta GmM)\right] - \right.$$
$$\left.\frac{R_0^3 e^{\frac{\beta GmM}{R_0}}}{3!}\left[\left(\frac{\beta GmM}{R_0}\right)^2 + \frac{\beta GmM}{R_0} + 2\right]\right\}. \tag{30}$$

For the mean energy, the result is (here $a = R_0$):

$$< \mathcal{U} >= -\frac{1}{3\beta\mathcal{Z}}2\sqrt{2}\pi^{5/2}\left(\frac{m}{x}\right)^{3/2}\Big(6a^3 e^{\frac{GmMx}{a}} - 3a^2 GmMxe^{\frac{GmMx}{a}} - a\mathrm{Gm}^2M^2x^2e^{\frac{GmMx}{a}} -$$
$$2aE_i((GmMx)/a)G^2m^2M^2x^2 - 2E_i((GmMx)/a)G\mathrm{G}m^2mM^3x^3 - 2G^3m^3M^3x^3 +$$
$$2G^3m^3M^3x^3e^{\frac{GmMx}{a}} + 3G^3m^3M^3x^3\mathrm{Ei}\left(\frac{GmMx}{a}\right) -$$
$$3G^3m^3M^3x^3\psi^{(0)}(z) + 3G^3m^3M^3x^3\log(GmMx)\Big). \tag{31}$$

For the specific heat, we have:

$$C_v = -\frac{1}{3axT\mathcal{Z}}\sqrt{2}\pi^{5/2}\left(\frac{m}{x}\right)^{3/2}\Big(-30a^4e^{\frac{GmMx}{a}} + 21a^3GmMxe^{\frac{GmMx}{a}} + a^2\mathrm{Gm}^2M^2x^2e^{\frac{GmMx}{a}} -$$
$$4a^2G^2m^2M^2x^2e^{\frac{GmMx}{a}} - 4aG\mathrm{G}m^2mM^3x^3e^{\frac{GmMx}{a}} -$$
$$8aG^3m^3M^3x^3 + 4aG^3m^3M^3x^3e^{\frac{GmMx}{a}} - 4G^2\mathrm{G}m^2m^2M^4x^4e^{\frac{GmMx}{a}} +$$
$$4G^4m^4M^4x^4e^{\frac{GmMx}{a}} + 3aG^3m^3M^3x^3\mathrm{Ei}\left(\frac{GmMx}{a}\right) -$$
$$3aG^3m^3M^3x^3\psi^{(0)}(z) + 3aG^3m^3M^3x^3\log(GmMx)\Big) +$$
$$\frac{\beta}{T}\left[\frac{1}{3\beta\mathcal{Z}}2\sqrt{2}\pi^{5/2}\left(\frac{m}{x}\right)^{3/2}\Big(6a^3e^{\frac{GmMx}{a}} - 3a^2GmMxe^{\frac{GmMx}{a}} - a\mathrm{Gm}^2M^2x^2e^{\frac{GmMx}{a}} -\right.$$
$$2aE_i((GmMx)/a)G^2m^2M^2x^2 - 2E_i((GmMx)/a)G\mathrm{G}m^2mM^3x^3 - 2G^3m^3M^3x^3 +$$

$$2G^3m^3M^3x^3e^{\frac{GmMx}{a}} + 3G^3m^3M^3x^3\mathrm{Ei}\left(\frac{GmMx}{a}\right) -$$
$$3G^3m^3M^3x^3\psi^{(0)}(z) + 3G^3m^3M^3x^3\log(GmMx)\Big)\Big]^2. \tag{32}$$

We plot the pertinent results below. Because of duality, our graphs closely resemble those of the precedent Section. See Figures 10–12.

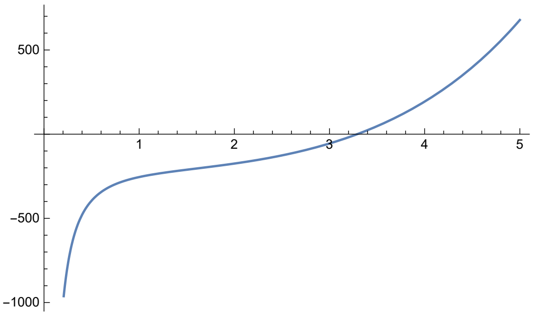

Figure 10. Plot of $\mathcal{Z}(x)$ for $x = \beta\ G = m = R_0 = M = 1$. Results turn out to be unphysical for $x < 3.35$, as in the previous case.

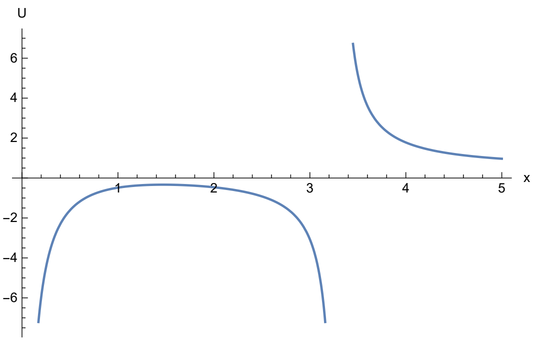

Figure 11. Plot of $< \mathcal{U}(x) >$ for $x = \beta$ and $G = m = R_1 = M = 1$. It closely resembles the companion graph of the precedent case. However, R_0 is too large to allow for statistical bounding in the region here considered.

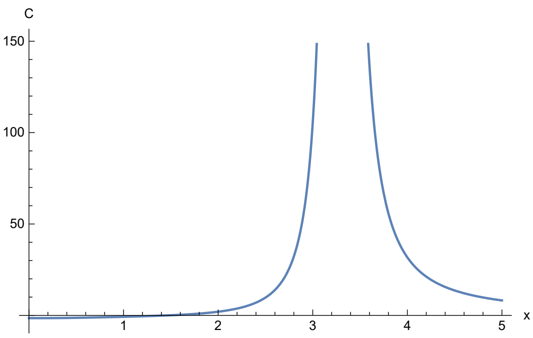

Figure 12. Plot of $C(x)$ for $x = \beta\ G = m = R_1 = M = 1$. $C > 0$ diverges in the unphysical $x-$region. It is positive in the physical one. The third law is complied with.

6. Conclusions

In this work, we showed how to deal with the partition function for gravitational systems in four different scenarios:

- The last two of the four scenarios envisioned here are linked by the twin transform from y to $1/y$.

- Even if our treatment is classical, the third law of thermodynamics is obeyed by the specific heat in all cases.
- It is remarkable that at a classical level, one can detect that at too high temperatures, statistical mechanics fails because the partition function becomes negative. We know now that at these Ts, matter cannot exist.
- Transformation from y to $1/y$: we might have discovered a transform that conserves the physics in the statistical mechanics of gravitation.

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References

1. Zamora, J.; Plastino, A.; Rocca, M.C.; Ferri, G.L. Dimensionally regularized Tsallis' Statistical Mechanics and two-body Newton's gravitation. *Physica A* **2018**, *497*, 310. [CrossRef]
2. Lynden-Bell, D.; Lynden-Bell, R.M. On the negative specific heat paradox. *Mon. Not. R. Astron. Soc.* **1977**, *181*, 405. [CrossRef]
3. Gradshteyn, S.; Ryzhik, I.M. *Table of Integrals, Series and Products*; Academic Press, Inc.: Cambridge, MA, USA, 1980.
4. Gel'fand, I.M.; Shilov, G.E. *Generalized Functions*; Academic Press: Cambridge, MA, USA, 1964; Volume 1.
5. Prudnikov, A.P.; Brichkov, Y.A.; Marichev, O.I. *Integrals and Series*; Gordon and Breach Science Publishers: London, UK, 1992.